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Weyl calculus and Noether currents: An application to cubic interactions

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Abstract

Cubic couplings between a complex scalar field and an infinite tower of symmetric tensor gauge fields are investigated. A symmetric conserved current, bilinear in a free scalar field and containing r derivatives, is provided for any rank $r \geqslant 1$ and is related to the corresponding rigid symmetry of Klein-Gordon's Lagrangian. Following Noether's method, the scalar field interacts with the tensor gauge fields via minimal coupling to the conserved currents. The corresponding cubic vertex is written in a very compact form by making use of Weyl's symbols. This enables the explicit computation of the nonAbelian gauge symmetry group, the elastic four-scalar scattering amplitude and the lower orders of the effective actions arising from integrating out either the scalar or the gauge fields.

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1 Two parallel Frønsdal programmes: deformation quantization and higher spins

By a suggestive coincidence, Frønsdal was involved during the same year (1978) in the birth of two seminal research programmes who prompted a large litterature over the last decades: the deformation quantization of Poisson manifolds [1] and the interaction problem of higher-spin gauge fields [2]. Although at first sight they seem to belong to distinct areas, say mathematical physics *versus* high-energy physics, the former has proved to be a decisive ingredient in the development of the latter. Incidentally, it is precisely in the construction of higher-spin (super)algebras [3] that star products made one of their earliest appearance in theoretical high-energy physics, a long time before noncommutative field theory. Actually, the underlying philosophy of Fedosov's quantization of symplectic manifolds [4] bears striking resemblances with Vasiliev's unfolded formulation of nonAbelian higher-spin gauge theories [5]. Further insights on the deep relationship between these latter constructions have been recently elaborated in [6] and might deserve further study. Indeed, the fruitful interplay between star products and higher-spin gauge field interactions has presumably not been fully uncovered yet. The present paper aims to provide another –but more elementary– instance of such connections between both subjects. The example provided here is also of pedagogical interest because it is a relatively simple –though nontrivial– application of very well-known techniques (Weyl and Wigner maps, Moyal product, *etc*) providing an original interpretation of various standard quantities (Weyl symbol, Wigner function, Moyal commutator, *etc*) in the specific context of higher-spin gauge theories.

The plan of the paper is as follows: Section 2 is a brief introduction to the celebrated formulation of quantum mechanics in terms of symbols (successively elaborated by Weyl, Wigner, Groenewold, Moyal, Berezin, and many others since then). Section 3 is a short review of the so-called Noether method for introducing consistent interactions with symmetric tensor gauge fields. Both formalisms are applied in Section 4 to the construction of cubic vertices, bilinear in a complex scalar field and linear in the gauge fields. (Similar ideas on the link between Weyl quantization and couplings between matter and gauge fields have been pushed forward previously in the context of conformal higher-spin theory by Segal [7].) Section 5 provides a general discussion of the action for the symmetric tensor gauge fields. In Section 6, the gauge invariant effective action induced by integrating out the scalar field is computed at lower orders. The residue of the propagator (determined in [8]) is discussed in Section 7 because, together with the cubic vertex, they constitute the key ingredients in the computation of the quartic interaction between complex scalars mediated by the infinite tower of gauge fields. Section 8 is devoted to the corresponding four-scalar elastic scattering amplitudes. The paper ends with a short conclusion in Section 9.

2 Weyl quantization

The main idea behind the deformation quantization programme [1] is the reformulation of the physical problem of quantization as the mathematical problem of deforming a commutative algebra of functions on a Poisson manifold into a noncommutative associative algebra.

In order to fix the ideas, one may consider the simplest case: good old quantum mechanics from our undergraduate studies. Classical mechanics is based on the commutative algebra of classical observables (*i.e.* real functions $f(x^\mu, p_\nu)$ on the phase space $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^{n*}$) endowed with the canonical Poisson bracket

$$\{f,g\} \,=\, \frac{\partial f}{\partial x^\mu}\,\frac{\partial g}{\partial p_\mu} \,-\, \frac{\partial f}{\partial p_\mu}\,\frac{\partial g}{\partial x^\mu}\,.$$

The *Weyl map* $\mathcal{W} : f(x^\mu, p_\nu) \mapsto \hat{F}(\hat{x}^\mu, \hat{p}_\nu)$ associates to any function f a Weyl(*i.e.* symmetric)-ordered operator $\hat{F}$ defined by

$$\hat{F}(\hat{x}^\mu, \hat{p}_\nu) \,=\, \frac{1}{(2\pi\hbar)^n}\int dk\, dv\;\mathcal{F}(k,v)\, e^{\frac{i}{\hbar}\,(k_\mu\,\hat{x}^\mu \,-\, v^\mu\,\hat{p}_\mu)}\,, \tag{1}$$

where $\mathcal{F}$ is the Fourier transform[1] of f over *whole* phase space (in other words, over position *and* momentum spaces)

$$\mathcal{F}(k,v) \,:=\, \frac{1}{(2\pi\hbar)^n}\int dx\, dp\;f(x,p)\, e^{-\frac{i}{\hbar}\,(k_\mu\,x^\mu \,-\, v^\mu\,p_\mu)}\,.$$

The function $f(x,p)$ is called the *Weyl symbol* of the operator $\hat{F}(\hat{x},\hat{p})$, which need not be in symmetric-ordered form. A nice property of the Weyl map (1) is that it relates the complex conjugation * of symbols to the Hermitian conjugation † of operators, $\mathcal{W} : f^*(x^\mu, p_\nu) \mapsto \hat{F}^\dagger(\hat{x}^\mu, \hat{p}_\nu)$. Consequently, the image of a real function (a classical observable) is a Hermitian operator (a quantum observable). The inverse $\mathcal{W}^{-1} : \hat{F}(\hat{x}^\mu, \hat{p}_\nu) \mapsto f(x^\mu, p_\nu)$ of the Weyl map is called the *Wigner map*.

The commutation relations between the position and momentum operators are $[\hat{x}^\mu, \hat{p}_\nu]_- = i\,\hbar\,\delta^\mu_\nu$, where $[\ ,\ ]_\pm$ denotes the (anti)commutator. The Baker-Campbell-Hausdorff formula implies that if

[1]The Weyl map is well defined for a much larger class than square integrable functions, including for instance the polynomial functions (remark: their Fourier transform are distributions).

the commutator $[\hat{X},\hat{Y}]_-$ itself commutes with both $\hat{X}$ and $\hat{Y}$, then

$$e^{\hat{X}}\,e^{\hat{Y}} = e^{\hat{X}+\hat{Y}+\frac{1}{2}[\hat{X},\hat{Y}]_-}\,.$$

Moreover, for any operators $\hat{X}$ and $\hat{Y}$ one can show that

$$e^{\hat{X}}\,e^{\hat{Y}}\,e^{\pm\hat{X}} = e^{[\hat{X},\;]_\pm}\,\hat{Y}\;,$$

where $[\hat{X},\;\,]_\pm$ denotes the (anti)adjoint action of $\hat{X}$. Two very useful equalities follow:

$$\begin{aligned} e^{\frac{i}{\hbar}\,(\,k_\mu\,\hat{x}^\mu\,-\,v^\mu\,\hat{p}_\mu)} &= e^{\frac{i}{2\hbar}\,k_\mu\,\hat{x}^\mu}\,e^{-\frac{i}{\hbar}\,v^\mu\,\hat{p}_\mu}\,e^{\frac{i}{2\hbar}\,k_\mu\,\hat{x}^\mu} \qquad (2)\\ &= e^{\frac{i}{2\hbar}\,k_\mu\,[\,\hat{x}^\mu,\;]_+}\,e^{-\frac{i}{\hbar}\,v^\mu\,\hat{p}_\mu} \qquad (3)\end{aligned}$$

Combining (1) with (3) implies that one way to explicitly perform the Weyl map is via some "anti-commutator ordering" for half of the variables with respect to their conjugates.

The matrix elements in the position basis of the exponential operator in (1) are found to be equal to

$$\begin{aligned} &\langle\, x \mid e^{\frac{i}{\hbar}\,(\,k_\mu\,\hat{x}^\mu\,-\,v^\mu\,\hat{p}_\mu)} \mid x'\,\rangle \\ &\quad = e^{\frac{i}{2\hbar}\,k_\mu(\,x^\mu + x'^{\,\mu})}\,\langle\, x \mid e^{-\frac{i}{\hbar}\,v^\mu\,\hat{p}_\mu} \mid x'\,\rangle \\ &\quad = \frac{1}{(2\pi\hbar)^n}\int dp\; e^{\frac{i}{2\hbar}\,k_\mu(\,x^\mu+x'^{\,\mu})\,+\,\frac{i}{\hbar}\,(\,x^\mu - x'^{\,\mu} - v^\mu)\,p_\mu} \qquad (4)\end{aligned}$$

by making use of the identity (2) and by inserting the completeness relation $\int dp \mid p\,\rangle\,\langle\, p \mid = \widehat{1}$.

The *integral kernel* of an operator $\hat{F}$ is the matrix element $\langle x \mid \hat{F} \mid x'\rangle$ appearing in the position representation of the state $\hat{F} \mid \psi\,\rangle$ as follows

$$\langle\, x \mid \hat{F} \mid \psi\,\rangle \;=\; \int dx'\;\psi(x')\;\langle\, x \mid \hat{F} \mid x'\,\rangle\,,$$

where the wave function in position space is $\psi(x') := \langle\, x' \mid \psi\,\rangle$ and the completeness relation $\int dx' \mid x'\,\rangle\,\langle\, x' \mid = \widehat{1}$ has been inserted. The definition (1) and the previous relation (4) enable to write the integral kernel of an operator in terms of its Weyl symbol,

$$\langle\, x \mid \hat{F} \mid x'\,\rangle \;=\; \frac{1}{(2\pi\hbar)^n}\int dp\; f\,\Big(\,\frac{x+x'}{2}\,,\,p\,\Big)\;e^{\frac{i}{\hbar}\,(\,x^\mu - x'^{\,\mu})\,p_\mu}\,. \qquad (5)$$

This provides an explicit form of the Wigner map

$$f(x^\mu,p_\nu) \;=\; \int dq\;\langle\, x - q/2 \mid \hat{F} \mid x + q/2\,\rangle\;e^{\frac{i}{\hbar}\,q^\mu\,p_\mu}\,, \qquad (6)$$

as follows from the expression (5). This shows that indeed the Weyl and Wigner maps are bijections between the vector spaces of classical and quantum observables. The Fourier transform

$$\breve{f}(x^\mu,v^\nu) \;:=\; \frac{1}{(2\pi\hbar)^n}\int dp\; f(x^\mu,p_\nu)\;e^{\frac{i}{\hbar}\,v^\mu\,p_\mu}\,,$$

over momentum space of the Weyl symbol $f(x,p)$ is a function on the configuration space $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. The equation (5) and (6) state that the Fourier transform over momentum space of the Weyl symbol is related to the integral kernel of its operator via

$$\langle\, x \mid \hat{F} \mid x'\,\rangle \;=\; \breve{f}\,\Big(\,\frac{x+x'}{2}\,,\,x^\mu - x'^{\,\mu}\,\Big) \qquad (7)$$

or, equivalently,

$$\breve{f}\big(x^\mu,v^\nu\big) \;=\; \langle\, x + v/2 \mid \hat{F} \mid x - v/2\,\rangle\,. \qquad (8)$$

By integrating over $x = x'$, the relation (5) also implies that the trace of an operator $\hat{F}$ is proportional to the integral over phase space of its Weyl symbol f,

$$\mathrm{Tr}[\hat{F}] \;=\; \frac{1}{(2\pi\hbar)^n}\int dx\,dp\;f(x,p)\,. \tag{9}$$

As a side remark, notice that the Fourier transform

$$\tilde{f}(k_\mu,p_\nu) \;:=\; \frac{1}{(2\pi\hbar)^n}\int dx\;f(x^\mu,p_\nu)\;e^{-\frac{i}{\hbar}k_\mu x^\mu}\,,$$

over position space of the Weyl symbol $f(x,p)$ is related to the matrix element in the momentum basis of the operator $\hat{F}(\hat{x},\hat{p})$ via

$$\langle\, k \mid \hat{F} \mid k' \,\rangle \;=\; \tilde{f}\left(\, k^\mu - k'^{\,\mu}\,,\; \frac{k+k'}{2}\,\right) \tag{10}$$

in direct analogy with (7).

The *Moyal product* $\star$ is the pull-back of the composition product in the algebra of quantum observables with respect to the Weyl map $\mathcal{W}$, such that the latter becomes an isomorphism of associative algebras, namely

$$\mathcal{W}\big[f(x,p)\,\star\,g(x,p)\big] \;=\; \hat{F}(\hat{x}^\mu,\hat{p}_\nu)\,\hat{G}(\hat{x}^\mu,\hat{p}_\nu)\,. \tag{11}$$

The Wigner map (6) allows to check that the following explicit expression of the Moyal product satisfies the definition (11),

$$\begin{aligned} f(x,p)\,\star\,g(x,p) \;&=\; f(x,p)\;\exp\left[\frac{i\,\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu}-\frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right]\;g(x,p)\\ &=\; f(x,p)\,g(x,p)+\frac{i\,\hbar}{2}\,\{f(x,p)\,,\,g(x,p)\}+\mathcal{O}(\hbar^2), \end{aligned} \tag{12}$$

where the arrows indicate on which factor the derivatives should act. The trace formula (9) for a product of operators leads to

$$\begin{aligned} \mathrm{Tr}[\,\hat{F}\hat{G}\,] \;&=\; \frac{1}{(2\pi\hbar)^n}\int dx\,dp\;f(x,p)\,\star\,g(x,p)\\ &=\; \frac{1}{(2\pi\hbar)^n}\int dx\,dp\;f(x,p)\,g(x,p) \end{aligned} \tag{13}$$

because all terms in the Moyal product (12) beyond the pointwise product are divergences over phase space and any boundary term will always be assumed to be zero in the present notes.

The *Wigner function* $\rho(x,p)$ is the Weyl symbol of the density operator $\hat{\rho}(\hat{x}^\mu,\hat{p}_\nu)$ under the Wigner map (6). Let $|\,\psi\,\rangle$ be an (unnormalized) quantum state. The corresponding pure state density operator is equal to $\hat{\rho} := |\,\psi\,\rangle\langle\,\psi\,|$. Then the Fourier transform over momentum space of the pure state Wigner function $\rho(x,p)$ can be written in terms of the wave function $\psi(x)$ as follows,

$$\breve{\rho}(x,q) = \psi(x+q/2)\,\psi^*(x-q/2)\,, \tag{14}$$

due to (8). The mean value of an observable $\hat{F}$ over the state $|\,\psi\,\rangle$ is proportional to the integral over phase space of the product between the Wigner function ρ and the Weyl symbol f,

$$\langle\, F\,\rangle_\psi \;=\; \frac{\langle\,\psi\mid\hat{F}\mid\psi\rangle}{\langle\,\psi\mid\psi\rangle} \;=\; \frac{\mathrm{Tr}[\,\hat{\rho}\,\hat{F}\,]}{\mathrm{Tr}\,[\,\hat{\rho}\,]} \;=\; \frac{\int dx\,dp\;\rho(x,p)\,f(x,p)}{\int dx\,dp\;\rho(x,p)}\,, \tag{15}$$

which explains why the Wigner function is sometimes called the Wigner "quasi-probability distribution." It should be emphasized that the Wigner function is real but may take negative values, thereby exhibiting quantum behaviour.

Let $\hat{H}(\hat{x},\hat{p})$ be a Hamiltonian operator of Weyl symbol $h(x,p)$. In the Heisenberg picture, the time evolution of quantum observables (which do not depend explicitly on time) is governed by the differential equation

$$\frac{d\hat{F}}{dt} \,=\, \frac{1}{i\,\hbar}\,[\hat{F},\hat{H}]_- \quad\Longleftrightarrow\quad \frac{df}{dt} \,=\, \frac{1}{i\,\hbar}\,[\,f \overset{\star}{,}\, h\,]_- \tag{16}$$

where $[\ \overset{\star}{,}\ ]_-$ denotes the *Moyal commutator* defined by

$$\begin{aligned}
[\,f(x,p) \overset{\star}{,}\, g(x,p)\,]_- &:= f(x,p)\,\star\,g(x,p)\,-\,g(x,p)\,\star\,f(x,p)\\
&= 2\,i\;f(x,p)\;\sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu}-\frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right]\,g(x,p)\\
&= i\,\hbar\,\{\,f(x,p)\,,\,g(x,p)\,\}\,+\,\mathcal{O}(\hbar^2)\,,
\end{aligned} \tag{17}$$

as can be seen from (12). If either $f(x,p)$ or $g(x,p)$ is a polynomial of degree two, then the Moyal commutator reduces to the first term in (17), *i.e.* essentially to the Poisson bracket. The *Moyal bracket* is the renormalization of the Moyal commutator given by $\frac{1}{i\,\hbar}[\ \overset{\star}{,}\ ]_- = \{\ ,\ \} + \mathcal{O}(\hbar)$. Since the Moyal bracket is a deformation of the Poisson bracket, one can see that the equation (16) in terms of the Weyl symbol is a perturbation of the Hamiltonian flow. When the Hamiltonian is quadratic (free) the quantum evolution of a Weyl symbol is identical to its classical evolution.

3 Noether method

A *symmetric conserved current* of rank $r \geqslant 1$ is a real contravariant symmetric tensor field $J^{\mu_1\cdots\mu_r}(x)$ obeying to the conservation law

$$\partial_{\mu_1}J^{\mu_1\cdots\mu_r}(x)\approx 0\,.$$

where the "weak equality" symbol $\approx$ stands for "equal on-mass-shell," *i.e.* modulo terms proportional to the Euler-Lagrange equations. A *generating function of conserved currents* is a real function $J(x,p)$ on phase space which is (i) a formal power series in the momenta and (ii) such that

$$\left(\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial x^\mu}\right)\,J(x,p)\approx 0\,. \tag{18}$$

This terminology follows from the fact that all the coefficients of order $r \geqslant 1$ in the power expansion of the generating function

$$J(x,p) \,=\, \sum_{r\geqslant 0}\frac{1}{r!}\,J^{\mu_1\cdots\mu_r}(x)\,p_{\mu_1}\dots p_{\mu_r} \tag{19}$$

are all symmetric conserved currents by means of (18).

A *symmetric tensor gauge field* of rank $r \geqslant 1$ is a real covariant symmetric tensor field $h_{\mu_1\dots\mu_r}(x)$ whose gauge transformations are [2]

$$\delta_\varepsilon h_{\mu_1\dots\mu_r}(x) \,=\, r\,\partial_{(\mu_1}\varepsilon_{\mu_2\dots\mu_r)}(x)\,+\,\mathcal{O}(h)\,, \tag{20}$$

where the gauge parameter $\varepsilon_{\mu_1\dots\mu_{r-1}}(x)$ is a covariant symmetric tensor field of rank $r-1$ and the round bracket denotes complete symmetrization with weight one. For lower ranks $r=1$ or 2, the transformation (20) either corresponds to the $U(1)$ gauge transformation of the vector ($r=1$) gauge field or to the linearized diffeomorphisms of the metric ($r=2$). This formulation of higher-spin gauge fields is sometimes called "metric-like" (in order to draw the distinction with the "frame-like" version where the gauge field is not completely symmetric) by comparison with the spin-two case. A *generating function of gauge fields* is a real function $h(x,v)$ on configuration space (i) which is a formal power series in the velocities and (ii) whose gauge transformations are

$$\delta_\varepsilon h(x,v) \,=\, \left(v^\mu\frac{\partial}{\partial x^\mu}\right)\,\varepsilon(x,v)\,+\,\mathcal{O}(h)\,, \tag{21}$$

where $\varepsilon(x,v)$ is also a formal power series in the velocities. The nomenclature follows from the fact that all the coefficients of order $r\geqslant 1$ in the power expansion of the generating function

$$h(x,v) \;=\; \sum_{r\geqslant 0}\frac{1}{r!}\,h_{\mu_1\ldots\mu_r}(x)\,v^{\mu_1}\ldots v^{\mu_r} \tag{22}$$

are all symmetric tensor gauge fields due to (21) with

$$\varepsilon(x,v) \;=\; \sum_{t\geqslant 0}\frac{1}{t!}\,\varepsilon_{\mu_1\ldots\mu_t}(x)\,v^{\mu_1}\ldots v^{\mu_t}\,.$$

The operator counting the rank (or "spin") r of the corresponding tensorial coefficients is denoted by

$$\hat{Y} \;:=\; v\cdot\frac{\partial}{\partial v}\,. \tag{23}$$

In the context of Noether couplings, the "velocities" v^μ and "momenta" p_ν are interpreted as mere auxiliary variables and can be assumed to be dimensionless. Accordingly, one sets $\hbar=\lambda$ from now on because this parameter should not be interpreted as Planck's constant but instead as a coupling constant with the dimension of a length. Let us introduce the nondegenerate bilinear pairing $\ll\,\|\,\gg$ between smooth functions $h(x,v)$ and $J(x,p)$ on the configuration and phase spaces respectively,

$$\ll h\,\|\,J\gg\;:=\;\int dx\;\exp\left(\frac{\partial}{\partial v^\mu}\frac{\partial}{\partial p_\mu}\right)h(x,v)\,J(x,p)\Big|_{v=p=0}\,. \tag{24}$$

If J and h are (formal) power series of the form (19) and (22) then the pairing (24) can be interpreted as the series

$$\ll h\,\|\,J\gg\;=\;\sum_{r\geqslant 0}\frac{1}{r!}\int dx\;h_{\mu_1\ldots\mu_r}(x)\,J^{\mu_1\ldots\mu_r}(x)\,. \tag{25}$$

Let us denote by $\ddagger$ the adjoint operation for the pairing (24) in the sense that

$$\ll \hat{\hat{O}}\,h\,\|\,J\gg\,=\,\ll h\,\|\,\hat{\hat{O}}^\ddagger\,J\gg\,,$$

where $\hat{\hat{O}}$ is an operator acting on the vector space of functions on configuration space (the double hat stands for "second quantization" in the sense that the operator $\hat{\hat{O}}$ acts on symbols of "first quantized" operators). Notice that $(v^\mu)^\ddagger=\partial/\partial p_\mu$ and $(\partial/\partial x^\mu)^\ddagger=-\partial/\partial x^\mu$ imply the useful relation

$$\left(v^\mu\frac{\partial}{\partial x^\mu}\right)^\ddagger \;=\; -\,\left(\frac{\partial}{\partial p_\mu}\,\frac{\partial}{\partial x^\mu}\right)\,. \tag{26}$$

Let $\breve{f}(x,q)$ be the Fourier transform of a function $f(x,p)$. Another useful property is that the pairing between this Fourier transform evaluated on the imaginary axis, $h(x,v):=\breve{f}(x,\frac{\lambda}{i}\,v)$, and a function $g(x,p)$ is proportional to the integral over phase space of the product of the functions f and g,

$$\begin{aligned}&\int dx\;\exp\left(\frac{\partial}{\partial v^\mu}\frac{\partial}{\partial p_\mu}\right)\breve{f}\left(x,\frac{\lambda}{i}\,v\right)\;g(x,p)\Big|_{v=p=0}\\ &=\;\frac{1}{(2\pi\lambda)^n}\int dx\,dp\;f(x,p)\,g(x,p)\,,\end{aligned} \tag{27}$$

due to the equalities

$$\begin{aligned}\exp\left(\frac{\partial}{\partial v^\mu}\frac{\partial}{\partial r_\mu}\right)g(x,r)\,e^{v^\mu p_\mu}\Big|_{v=r=0} \;&=\; g(x,\partial/\partial v)\,e^{v^\mu p_\mu}\,\big|_{v=0}\\ &=\; g(x,p)\,.\end{aligned}$$

The *matter action* is a functional $S_0[\phi]$ of some matter fields collectively denoted by ϕ . The Noether method for introducing interactions is essentially the "minimal" coupling between a gauge field $h_{\mu_1\ldots\mu_r}(x)$ and a conserved current $J^{\mu_1\cdots\mu_r}[\,\phi(x)\,]$ of the same rank. The *Noether interaction* S_1 between gauge fields and conserved currents is defined as the pairing between their generating functions

$$S_1[\phi,h] \,:=\, \ll h\,\|\,J \gg \,=\, \sum_{r\geqslant 0}\frac{1}{r!}\int dx\; h_{\mu_1\ldots\mu_r}(x)\,J^{\mu_1\cdots\mu_r}(x)\,, \tag{28}$$

where (25) has been used. Let us assume that there exists a gauge invariant action $S[\phi,h]$ whose power expansion in the gauge fields starts as follows

$$S\,[\phi,h] \;=\; S_0[\phi] \,+\, S_1[\phi,h] \,+\, S_2[\phi,h] \,+\, \mathcal{O}(h^3)\,. \tag{29}$$

The gauge variation of the Noether interaction (28) under (21)

$$\delta_\varepsilon S_1[\phi,h] \;=\, \ll \delta_\varepsilon h\,\|\,J \gg \,+\,\mathcal{O}(h)\,,$$

is at least of order one in the gauge fields when the equations of motion for the matter sector are obeyed,

$$\delta_\varepsilon S_1[\phi,h] \;\approx\; \mathcal{O}(h)\,. \tag{30}$$

Indeed, the properties (18) and (26) imply that

$$\ll \Big(v\cdot\frac{\partial}{\partial x}\Big)\,\varepsilon\,\|\,J \gg =\; -\; \ll \varepsilon\,\|\,\Big(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial x}\Big)\,J \gg \,\approx\, 0\,, \tag{31}$$

which is presumably more familiar in components. The equation (30) implies that the action (29) might indeed be gauge-invariant at lowest order in the gauge fields because the terms that are proportional to the Euler-Lagrange equations $\delta S_0/\delta\phi$ of the matter sector could be compensated by introducing a gauge transformation $\delta_\varepsilon\phi$ of the matter fields, linear in the gauge parameters ε and in the matter fields ϕ , such that

$$\delta_\varepsilon\Big(\,S_0[\phi] + S_1[\phi,h]\,\Big) \;=\; \mathcal{O}(h)\,. \tag{32}$$

A *Killing tensor field* of rank $r-1\geqslant 0$ on $\mathbb{R}^n$ is a covariant symmetric tensor field $\bar\varepsilon_{\mu_1\ldots\mu_{r-1}}(x)$ solution of the generalized Killing equation

$$\partial_{(\mu_1}\bar\varepsilon_{\mu_2\ldots\mu_r)}(x) = 0\,.$$

A *generating function of Killing fields* is a function $\bar\varepsilon(x,v)$ on configuration space which is (i) a formal power series in the velocities and (ii) such that $\bar\varepsilon(x+v\,\tau\,,v)=\bar\varepsilon(x,v)$ for any τ . Then the coefficients in the power series

$$\bar\varepsilon(x,v) \;=\; \sum_{t\geqslant 0}\frac{1}{t!}\,\bar\varepsilon_{\mu_1\ldots\mu_r}(x)\,v^{\mu_1}\ldots v^{\mu_t}$$

are all Killing tensor fields on $\mathbb{R}^n$. The variation (20) of the gauge field vanishes if the gauge parameter is a Killing tensor field. Therefore the corresponding gauge transformation $\delta_{\bar\varepsilon}\phi$ of the matter fields is a rigid symmetry of the matter action $S_0[\phi]$ since

$$\delta_{\bar\varepsilon}S_0[\phi] = -\delta_{\bar\varepsilon}S_1[\phi,h]\,\Big|_{h=0} = 0\,,$$

due to (32) and the fact that $\delta_\varepsilon\phi$ is independent of the gauge fields. In turn, this shows that the conserved current $J^{\mu_1\cdots\mu_r}[\,\phi(x)\,]$ must be equal, on-shell and modulo a trivial conserved current (sometimes called an "improvement"), to the Noether current associated with the latter rigid symmetry of the action $S_0[\phi]$. Killing tensor fields on flat spacetime and their link with higher-spin gauge theories were discussed in more details in [9] and references therein.

4 Application to the scalar field

Consider a matter sector made of a free complex scalar field ϕ, of mass square $M^2 \in \mathbb{R}$, propagating on Minkowski spacetime with mostly plus metric $\eta_{\mu\nu}$. The matter action is the quadratic functional

$$S_0[\phi] \,=\, -\frac{\lambda^2}{2}\int dx \left(\,|\,\partial_\mu\phi(x)|^2 + M^2|\,\phi(x)|^2\,\right) \,=\, -\frac{1}{2}\,\langle\,\phi\mid\,\hat{p}^2+m^2\,\mid\phi\rangle \tag{33}$$

where the parameter $m := \lambda\, M$ is dimensionless and the operator $\hat{p}^2 := \eta^{\mu\nu}\hat{p}_\mu\hat{p}_\nu = -\,\lambda^2\,\Box$ is related to the wave operator. The Klein-Gordon action $S_0[\phi]$ is thus proportional to the mean value over the state $|\,\phi\,\rangle$ of the Hamiltonian (constraint)

$$\hat{H}_0 := \frac{1}{2}\,(\,\hat{p}^2+m^2\,)$$

of a free relativistic particle. The Klein-Gordon equation reads as

$$(\,\hat{p}^2+m^2\,)\,|\,\phi\,\rangle \approx 0\,.$$

The Minkowski metric provides an isomorphism between the tangent and cotangent spaces via the identification $p_\mu = \eta_{\mu\nu}v^\nu$, which induces an isomorphism between the spaces of functions on the configuration and phase spaces. By a slight abuse of notation but for the sake of simplicity, the function $f(x^\mu,p_\nu)$ and the function

$$g(x^\mu,v^\nu) := f(x^\mu,p_\nu=\eta_{\nu\sigma}v^\sigma)$$

will be identified and denoted from now on by the same symbol f but with different arguments, respectively $f(x,p)$ and $f(x,v)$. Following the identitification between the phase and configuration space, one finds that a very simple generating function of conserved currents is $J(x,v)=\check{\rho}(x,\frac{\lambda}{i}v)$, which is the analytic continuation of the Fourier transform over momentum space of the pure state Wigner function $\rho(x,p)$ for the state $|\,\phi\,\rangle$. More pragmatically, it can be written in terms of the wave function $\phi(x)$ as follows,

$$J(x,v) = \phi(\,x - i\,\lambda\,v/2\,)\,\phi^*(\,x + i\,\lambda\,v/2\,) = |\,\phi(\,x - i\,\lambda\,v/2\,)\,|^2\,, \tag{34}$$

due to (14) and $\hat{\rho} = |\,\phi\,\rangle\langle\,\phi\,|$. Notice that the generating function (34) is manifestly real. The condition (18) can then be checked by direct computation. Moreover, the Taylor expansion of (14) in power series of the velocities leads to the explicit expression of the symmetric conserved currents

$$J_{\mu_1\ldots\mu_r}(x) \,=\, \left(\frac{i\,\lambda}{2}\right)^r\sum_{s=0}^{r}(-1)^s\begin{pmatrix} r\\ s\end{pmatrix}\partial_{(\mu_1}\ldots\partial_{\mu_s}\phi(x)\;\partial_{\mu_{s+1}}\ldots\partial_{\mu_r)}\phi^*(x). \tag{35}$$

The symmetric conserved current (35) or rank r is bilinear in the scalar field and contains exactly r derivatives. The currents of any rank are real thus, if the scalar field is real then the odd rank currents are absent due to the factor in front of (35). Analogous explicit sets of conserved currents were already provided in [10]. Notice that the symmetric conserved current of rank two

$$J_{\mu\nu}(x) \,=\, -\,\frac{\lambda^2}{4}\Big(\partial_\mu\partial_\nu\phi(x)\;\phi^*(x)\,+\,\phi(x)\;\partial_\mu\partial_\nu\phi^*(x)\,-\,2\,\partial_{(\mu}\phi(x)\;\partial_{\nu)}\phi^*(x)\Big)$$

is distinct from the canonical energy-momentum tensor

$$T_{\mu\nu}(x) \,=\, \frac{\lambda^2}{2}\Big[\,2\,\partial_{(\mu}\phi(x)\;\partial_{\nu)}\phi^*(x)\,-\,\eta_{\mu\nu}\Big(\,|\,\partial_\rho\phi(x)|^2 + M^2|\,\phi(x)|^2\,\Big)\Big]$$

though, on-shell they differ only from a trivially conserved current since

$$J_{\mu\nu}(x) \,\approx\, T_{\mu\nu}(x) \,+\, \frac{\lambda^2}{4}\,\big(\,\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\big)|\,\phi(x)|^2\,. \tag{36}$$

By virtue of (15) and (27), the Noether interaction (28) defined by the generating function (34) can be written as the "mean value" over the state $|\,\phi\,\rangle$ of the image $\hat{H}(\hat{x},\hat{p})$ of the generating function $h(x,p)$ under the Weyl map (1),

$$S_1[\phi,h] \,=\, \langle\,\phi\,|\,\hat{H}\,|\,\phi\rangle\,. \tag{37}$$

Similar Noether interactions with scalar field conserved currents were elaborated in [11, 12]. By making use of the "anticommutator ordering" prescription for the Weyl map, as explained in Section 2, one finds that the operator $\hat{H}$ starts at lower spin as

$$\begin{aligned}\hat{H}(\hat{x},\hat{p}) &= h(x)+\frac{1}{2}\Big(\hat{p}_\mu h^\mu(x)+h^\mu(x)\,\hat{p}_\mu\Big)\\ &\quad+\frac{1}{8}\Big(\hat{p}_\mu\hat{p}_\nu\,h^{\mu\nu}(x)+2\,\hat{p}_\mu h^{\mu\nu}(x)\,\hat{p}_\nu+h^{\mu\nu}(x)\,\hat{p}_\mu\hat{p}_\nu\Big)\,+\,\ldots\end{aligned}$$

As one can check, the Noether coupling with the vector gauge field h_μ is the usual electromagnetic coupling. The Noether coupling with the symmetric tensor gauge field $h_{\mu\nu}$ corresponds to the "minimal" coupling between a spin-two gauge field and a scalar density ϕ of weight one-half (minimal in the sense that there is no term containing the trace $\eta^{\mu\nu}h_{\mu\nu}$ corresponding to the (linearized) volume element in the interaction). This means that $|\phi|^2$ must be a density of weight one. As can be checked directly from (36), the action (29) reads

$$\begin{aligned}S[\phi,h] &= -\frac{\lambda^2}{2}\int dx\,(-g)^{\frac{1}{2}}\Big[g^{\mu\nu}\,\partial_\mu\Phi(x)\,\partial_\nu\Phi^*(x)+\Big(\frac{R}{4}-M^2\Big)|\,\Phi(x)|^2\Big]\\ &\quad+\mathcal{O}(h^2)\,,\end{aligned} \tag{38}$$

in terms of the scalar $\Phi := (-g)^{-\frac{1}{4}}\phi$, the metric $g_{\mu\nu} := \eta_{\mu\nu}+h_{\mu\nu}+\mathcal{O}(h^2)$ and the scalar curvature R. It is worth emphasizing that the cubic interaction $\int dx\; h_{\mu_1\ldots\mu_r}\,J^{\mu_1\cdots\mu_r}$ contains r derivatives and grows like the power $r-3+n/2$ of the energy scale by naive dimensional analysis, so if it involves a tensor field of rank $r>3-n/2$ then it is not (power-counting) renormalizable. Notice also that for a real scalar field, the interactions occur with tensor gauge fields of even rank only.

The quadratic and cubic functionals (33) and (37) are such that the would-be action (29) at all orders in the gauge fields starts as

$$S\,[\phi,h] \,=\, -\,\langle\,\phi\,|\,\hat{G}\,|\,\phi\rangle\,+\,\mathcal{O}(\phi^3,h^2)\,, \tag{39}$$

where the operator

$$\hat{G} \,:=\, \hat{H}_0\,-\,\hat{H} \tag{40}$$

should be interpreted in terms of its Weyl symbol

$$g(x,p) \,=\, h_0(x,p)\,-\,h(x,p)$$

as the generating function of the various gauge fields around Minkowski metric as background, $h_0(x,p) := \frac{1}{2}\,(p^2+m^2)$. The linearized gauge transformation (21) of the Weyl symbol $h(x,p)$ can be written as the Poisson bracket between the function $\varepsilon(x,p)$ and the Weyl symbol of the Hamiltonian $h_0(x,p)$ of a free relativistic particle,

$$\Big(p^\mu\frac{\partial}{\partial x^\mu}\Big)\,\varepsilon(x,p) \,=\, \{\,\varepsilon(x,p)\,,\,h_0(x,p)\,\} \,=\, \frac{1}{i\,\lambda}\,[\,\varepsilon(x,p)\,\star\,h_0(x,p)\,]_-\,. \tag{41}$$

The image of (41) under the Weyl map leads to

$$\delta_{\hat{\varepsilon}}\hat{H} \,=\, \frac{1}{i\,\lambda}\,[\,\hat{\varepsilon},\hat{H}_0\,]_-\,+\,\mathcal{O}(\hat{H})\,. \tag{42}$$

The variation of the scalar field ϕ which guarantees the gauge invariance, at lowest order in h, of the action (39) is

$$\delta_{\hat{\varepsilon}}\,|\,\phi\,\rangle \,=\, -\,\frac{1}{i\,\lambda}\,\hat{\varepsilon}\,|\,\phi\,\rangle\,, \tag{43}$$

as can be checked directly. At lower orders in the derivative, the explicit form of the operator $\hat{\varepsilon}(\hat{x},\hat{p})$ in terms of its Weyl symbol $\varepsilon(x,p)$

$$\hat{\varepsilon}(\hat{x},\hat{p}) \;=\; \varepsilon(x) + \underbrace{\frac{1}{2}\Big(\hat{p}_\mu\varepsilon^\mu(x) + \varepsilon^\mu(x)\,\hat{p}_\mu\Big)}_{=\,\varepsilon^\mu\,\hat{p}_\mu + \frac{1}{2}(\hat{p}_\mu\varepsilon^\mu)} + \ldots$$

confirms that following (43) the matter field ϕ transforms as a scalar density of weight one-half under the (linearized) diffeomorphisms. The set of all such transformations (43) closes under the Moyal bracket and is isomorphic to the Lie algebra of Hermitian operators, *i.e.* the Lie algebra of quantum observables. If one truncates the tower of gauge fields to the lower-spin sector ($r \leqslant 2$), then the Lie subalgebra of symmetries one is left with is the direct sum of the local $\mathfrak{u}(1)$ algebra and the vector field algebra. The form of (39) suggests the following finite gauge transformation

$$|\,\phi\,\rangle \longrightarrow \hat{U}\,|\,\phi\,\rangle\,, \quad \hat{G} \longrightarrow \hat{U}\,\hat{G}\,\hat{U}^{-1}\,, \quad \text{with} \quad \hat{U} := \exp\left(\frac{i}{\lambda}\,\hat{\varepsilon}\right) \tag{44}$$

because, *at lowest order* in $\hat{H}$, it reproduces the infinitesimal transformations (42)-(43) and leaves invariant the quadratic form $\langle\,\phi\,|\,\hat{G}\,|\,\phi\rangle$. The scalar and gauge fields respectively transforms in the fundamental and adjoint representation of the group of unitary operators. Notice that as long as higher-derivative transformations are allowed then the infinite tower of higher-spin fields should be included for consistency of the gauge transformations (44) beyond the lowest order. The infinitesimal version of (44) written in terms of the Weyl symbols leads to the following completion of (21)

$$\begin{aligned}\delta_\varepsilon h(x,p) \;&=\; \frac{1}{i\,\lambda}\Big[\,\varepsilon(x,p) \;\overset{\star}{,}\; g(x,p)\Big]_- \\ &= \left(\eta^{\mu\nu}p_\mu\frac{\overrightarrow{\partial}}{\partial x^\mu} + \frac{2}{\lambda}\,h(x,p)\;\sin\left[\frac{\lambda}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right)\right]\right)\varepsilon(x,p)\end{aligned}$$

where one made use of (17) and (41).

The Weyl symbol $\overline{\varepsilon}(x,p)$ of an operator $\hat{\overline{\varepsilon}}(\hat{x},\hat{p})$ commuting with the Hamiltonian $\hat{H}_0$ of the free relativistic particle is a generating function of Killing fields, as can be seen easily from (41). This is in agreement with the facts that if $[\,\hat{\overline{\varepsilon}}\,,\hat{H}_0\,]_- = 0$ then the corresponding transformation (44),

$$|\,\phi\,\rangle \longrightarrow \exp\left(\,i\,\hat{\overline{\varepsilon}}\,/\lambda\,\right)\,|\,\phi\,\rangle, \tag{45}$$

is obviously a symmetry of the Klein-Gordon action (33) and the corresponding variation (44) vanishes, $\delta_{\hat{\overline{\varepsilon}}}\hat{H} = 0$. It is very tempting to conjecture that the full action (39) should be interpreted as arising from the gauging of the rigid symmetries (45) of the free scalar field, which generalize the $U(1)$ and Poincaré symmetries, so the local symmetries (44) generalize the local $U(1)$ and diffeomorphisms. The rigid higher-derivative symmetries which are generated by Hermitian operators $\hat{\overline{\varepsilon}}(\hat{p})$ independent of the position and which thereby generalize the phase shifts and translations were introduced in [13]. Notice that the conserved currents (35) are indeed equivalent to the Noether currents for the latter symmetries, as can be checked by direct computation. The group of unitary opeators was already advertised in [11] as the symmetry group arising from the gauging of these rigid higher-derivative symmetries. In the case of a real scalar field, the Lie algebra and group of gauge symmeties would have to be replaced by, respectively, the algebra of symmetric operators and the group of orthogonal operators. The former construction works along the same line for a scalar field taking values in an internal finite-dimensional space, *i.e.* for a multiplet of scalar fields.

5 Symmetric tensor gauge field action

On Minkowski spacetime, the pairing (24) can be interpreted as a non-degenerate symmetric bilinear form over the vector space of smooth real functions $h(x,v)$ on $\mathbb{R}^{2n}$,

$$\begin{aligned} \ll h \,\|\, h' \gg \;&=\; \int dx\, \exp\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v'}\right) h(x,v)\, h'(x,v')\Big|_{v=v'=0}\,, \\ &=\; (2\pi\lambda)^n \int dk\, \exp\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v'}\right) \tilde{h}(k,v)\, \tilde{h}'(-k,v')\Big|_{v=v'=0}\,, \end{aligned}$$

where the dot stands for the contraction of indices by making use of the Minkowski metric. Endowed with the above-mentioned bilinear form, it is very suggestive to interpret $h\big(x^\mu,v^\nu\big)$ as a string field $h\big(x^\mu,(a^\nu)^\dagger\big)|\,0\rangle$ from the leading Regge trajectory. In such case, the vacuum state is identified with the unit function $|\,0\rangle \leftrightarrow \|\,1\gg$ and the creation/destruction operators with $v^\nu \leftrightarrow (a^\nu)^\dagger$ and $\partial/\partial v^\mu \leftrightarrow \eta_{\mu\nu}\, a^\nu$. Therefore the Fock space $\{f(a^\dagger)|\,0\rangle\}$ corresponds to the space $\{f(v)\}$ of real polynomial functions of the "velocities" only.

For an operator $\hat{F}(v^\mu,\partial/\partial v^\nu)$ acting only on the Fock space, the normal ordering is rather natural. The *normal symbol* $f(v^\mu,u^\nu)$ of the operator $\hat{F}$ can be defined by

$$f(v^\mu,u^\nu) \;:=\; e^{-u\cdot v}\,\hat{F}\left(v,\frac{\partial}{\partial v}\right)\, e^{+\,u\cdot v}\,. \tag{46}$$

The *normal map* $\mathcal{N}: f(v^\mu,u^\nu)\mapsto \hat{F}(v^\mu,\partial/\partial v^\nu)$ associates to any symbol f a normal-ordered operator $\hat{F}$ such that (46). A nice feature of the normal map is that $\mathcal{N}\,:\, f^*(u^\mu,v^\nu)\;\mapsto\;\big(\hat{F}(v^\nu,\partial/\partial v^\mu)\big)^\dagger$. Consequently, the image of a real and symmetric (*i.e.* such that $f(u^\mu,v^\nu)=f(v^\mu,u^\nu)$) function is a Hermitian operator. The *normal product* $\star_{_N}$ is the pull-back of the composition product with respect to the normal map $\mathcal{N}$ such that the latter becomes an isomorphism of associative algebras, namely

$$\mathcal{N}\big[f(v,u)\,\star_{_N}\, g(v,u)\big] \;=\; \hat{F}(v,\partial/\partial v)\,\hat{G}(v,\partial/\partial v)\,.$$

One can show that

$$f(v,u)\,\star_{_N}\, g(v,u) \;=\; f(v,u)\,\exp\left(\frac{\overleftarrow{\partial}}{\partial u}\cdot\frac{\overrightarrow{\partial}}{\partial v}\right) g(v,u)\,. \tag{47}$$

where the arrows indicate on which factor the derivatives should act. The interest of the normal symbol f of the operator $\hat{F}$ acting on Fock space for the present paper is the identity

$$\begin{aligned} &\ll h \,\|\, \hat{F} \,\|\, h' \gg = \\ &= \int dx\, \exp\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v'}\right) f\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v'}\right) h(x,v)\, h'(x,v')\Big|_{v=v'=0}\,, \end{aligned} \tag{48}$$

which comes from $(v')^\ddagger=\partial/\partial v$.

Two identities which will be useful for later purpose follow from the property that the Fourier transform of a Gaussian is also a Gaussian. For instance,

$$\int dx\, dp\; e^{-\beta p^2/2}\, h(x,p) \;=\; \left(\frac{2\pi}{\beta}\right)^{\frac{n}{2}} \int dx\;\; e^{\frac{1}{2\beta}\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\right)}\, h(x,v)\Big|_{v=0} \tag{49}$$

is a consequence of (27). Moreover, the Leibnitz rule implies that

$$\frac{\partial}{\partial v}\big[\, h(x,v)\, h'(x,v)\,\big] = \left(\frac{\partial}{\partial v}+\frac{\partial}{\partial v'}\right)\,\big[h(x,v)\, h'(x,v')\big]\,\Big|_{v=v'}$$

thus (49) leads to

$$\int dx\, dp\; e^{-\beta p^2/2}\, h(x,p)\, h'(x,p) \;=\; \left(\frac{2\pi}{\beta}\right)^{\frac{n}{2}} \ll \check{h}\,\|\,\check{h}'\gg\,, \tag{50}$$

where the "checked" functions have been defined by

$$\check{h}(x,v\,;\beta) \;:=\; \exp\left[\frac{1}{2}\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\right)\right]\, h\big(x,p=\beta^{-\frac{1}{2}}\,v\big)\,. \tag{51}$$

In components, (51) reads

$$\begin{aligned}&\check{h}_{\mu_1\ldots\mu_r}(x\,;\beta)\\ &=\sum_{m\geqslant 0}\frac{\beta^{-(m+\frac{r}{2})}}{2^m\, m!\begin{pmatrix}2m+r\\ r\end{pmatrix}}\;\eta^{\nu_1\nu_2}\ldots\eta^{\nu_{2m-1}\nu_{2m}}h_{\mu_1\ldots\mu_r\nu_1\ldots\nu_{2m}}(x)\,.\end{aligned}$$

The *gauge action* is the restriction $S[\phi=0,h]$ of the action (29) to the gauge sector only. The actions on Minkowski spacetime presented in [2, 8] are quadratic functionals of the gauge fields only taking the form

$$\begin{aligned} S_2\,[\,\phi=0\,,\,h\,] \;&:=\; -\frac{1}{2}\;\ll h\,\|\,\hat{\hat{K}}\,\|\,h\gg \\ &=\; -\sum_{r\geqslant 0}\frac{1}{2\,r!}\int dx\; h^{\mu_1\ldots\mu_r}(x)\,\big(\hat{\hat{K}}h\big)_{\mu_1\ldots\mu_r}(x)\,,\end{aligned} \tag{52}$$

where $\hat{\hat{K}}$ is the *kinetic operator*. The latter is an operator acting over the functions $h(x,v)$ on the configuration space which (i) is self-adjoint, $\hat{\hat{K}}^{\ddagger}=\hat{\hat{K}}$, and (ii) takes the form

$$\hat{\hat{K}} = \hat{M}\left[\,\hat{p}^2+(v\cdot\hat{p})\,\hat{\hat{S}}\,\right]\,, \tag{53}$$

where $\hat{\hat{S}}$ is some known local operator the specific form of which plays no important role in the present discussion and $\hat{M}(v,\partial/\partial v)$ is a self-adjoint operator which acts only on the Fock space. More precisely, it takes the form of a power series

$$\hat{M}(v,\partial/\partial v) \;=\; \sum_{m\geqslant 0} M_m\Big(v\cdot\frac{\partial}{\partial v}\Big)\;\Big(v\cdot v\Big)^m\Big(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\Big)^m\,, \tag{54}$$

where the coefficients $M_m(y)$ are rational functions of a single variable.

6 Induced gauge action

Following the philosophy of Sakharov's induced gravity, one may focus on the gauge effective action arising from the minimal coupling of the matter field to the gauge fields without any gauge action initially. The kinetic terms for the gauge fields are then generated by integrating out the matter field. In the present case, the path integral for the quadratic functional $\langle\,\phi\mid\hat{G}\mid\phi\,\rangle$ is proportional to the determinant of the kinetic operator $\hat{G}$

$$\int\mathcal{D}\phi\, e^{-\langle\phi|\hat{G}\,|\phi\rangle}\;\propto\;\det\,[\hat{G}\,]^{-1}\;=\;e^{-\log\det\,[\hat{G}]}$$

The one-loop effective action for the gauge fields

$$S_{eff}[g] \;:=\; \log\det\,[\hat{G}\,] \;=\; \mathrm{Tr}\,[\,\log\hat{G}\,]$$

can be computed by making use of the identity

$$\log b\;-\;\log a\;=\;\int_0^\infty\frac{d\beta}{\beta}\,\Big(e^{-\,a\,\beta}-e^{-\,b\,\beta}\Big)\,. \tag{55}$$

More precisely, the difference between the one-loop effective actions for the generating function $g(x,p)$ and the (infinite) contribution for the background $h_0(x,p)$ is equal to

$$S_{eff}[g]-S_{eff}[h_0]=-\int_0^\infty\frac{d\beta}{\beta}\,\mathrm{Tr}\,\Big[e^{-\beta\hat{G}}-e^{-\beta\hat{H}_0}\Big]\,. \tag{56}$$

Notice that (56) might still need infrared and ultraviolet regularizations. Let us consider the operator (40) as a perturbed Hamiltonian with $\hat{H}$ as interaction Hamiltonian and work in the Dirac picture:

$$e^{-\beta\hat{G}}\;=\;\hat{U}_0(\beta)\;\mathrm{T}\,e^{\int\limits_0^\beta d\tau\,\hat{H}(\tau)}\,, \tag{57}$$

where $\hat{U}_0(\tau):=e^{-\tau\hat{H}_0}$ is the evolution operator for the free relativistic particle, while the evolved interaction Hamiltonian is

$$\hat{H}(\tau)\;:=\;\hat{U}_0^{-1}(\tau)\,\hat{H}\,\hat{U}_0(\tau)\;=\;\hat{H}(\,\hat{x}^\mu\;-\;i\,\lambda\,\tau\hat{p}^\mu\,,\,\hat{p}_\nu) \tag{58}$$

and the symbol "T" in (57) stands for the time ordering. Thus the expansion (56) in power series of the perturbation $h(x,p)$ starts as

$$\begin{aligned}
&S_{eff}[g]-S_{eff}[h_0]\\
&=-\int_0^\infty\frac{d\beta}{\beta}\,e^{-\beta\,m^2/2}\Big(\int\limits_0^\beta d\tau\;\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}(\tau)\Big]\;+\\
&\quad+\int\limits_0^\beta d\tau_1\int\limits_0^{\tau_1}d\tau_2\;\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}(\tau_1)\,\hat{H}(\tau_2)\Big]\Big)\;+\;\mathcal{O}(h^3)\,,\\
&=-\lim_{\Lambda\to\infty}\int_{1/\Lambda}^\infty d\beta\,e^{-\beta\,m^2/2}\Big(\;\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}\Big]\;+\\
&\quad+\int\limits_0^\beta d\tau\big(1-\frac{\tau}{\beta}\big)\;\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}(\tau)\,\hat{H}\Big]\Big)\;+\;\mathcal{O}(h^3)\,,
\end{aligned} \tag{59}$$

where the mass square obviously plays the role of an infrared regulator and Λ has been introduced as an ultraviolet cutoff. The traces in (59) can be written explicitly as local integrals over spacetime by making use of (12), (13) and (58). For instance, the trace in the linear term reads explicitly as

$$\begin{aligned}
&\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}\Big]=\frac{1}{(2\pi\lambda)^n}\int dx\,dp\;e^{-\beta p^2/2}\,h(x,p)\\
&=\frac{1}{(2\pi\lambda^2\beta)^{\frac{n}{2}}}\int dx\;\;e^{\frac{1}{2\beta}(\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p})}\,h(x,p)\Big|_{p=0}\\
&=\frac{1}{(2\pi\lambda^2\beta)^{\frac{n}{2}}}\sum_{m\geqslant 0}\frac{(2m)!}{m!\,(2\beta)^m}\int dx\;\eta^{\mu_1\mu_2}\ldots\eta^{\mu_{2m-1}\mu_{2m}}h_{\mu_1\ldots\mu_{2m}}(x)\,,
\end{aligned}$$

where (27) and (49) have been used. The linear term of the one-loop effective action therefore corresponds to a (linearized) "cosmological" term for each even-spin gauge field. Alternatively, this term can be expressed more concisely in terms of the field redefinition $h(x,p)\to\check{h}(x,p)$ with (51) since it is proportional to the spacetime integral of the scalar function $\check{h}(x,p=0)$.

In order to obtain concrete expressions for the quadratic term of the one-loop effective action, it is useful to insert the completeness relation $\int dk\mid k\,\rangle\,\langle\,k\mid=\widehat{1}$ between each operator and apply the identity (10),

$$\begin{aligned}
&\mathrm{Tr}\,\Big[e^{-\beta\hat{p}^2/2}\,\hat{H}(\tau)\,\hat{H}\Big]\\
&=\int dk\,dk'\;e^{-\beta k^2/2}\,e^{\tau(k^2-k'^2)/2}\,\Big|\,\tilde{h}\big(k-k',\frac{k+k'}{2}\big)\,\Big|^2\\
&=\int d\ell\,d\ell'\;e^{-\frac{\beta}{2}(\frac{\ell}{2}+\ell')^2}\,e^{\tau(\ell\cdot\ell')}\,\Big|\,\tilde{h}\big(\ell,\ell'\big)\,\Big|^2\,.
\end{aligned}$$

This leads to

$$
\int_0^\beta d\tau \big(1 - \frac{\tau}{\beta} \big) \ \mathrm{Tr} \left[e^{-\beta \hat{p}^2/2} \, \hat{H}(\tau) \, \hat{H} \, \right] \Big)
$$

$$
\begin{aligned}
&= \int d\ell \, d\ell' \ e^{-\frac{\beta}{2}(\frac{\ell}{2}+\ell')^2} \, \frac{1}{(\ell \cdot \ell')^2} \left(e^{\beta(\ell\cdot\ell')} - 1 - \beta(\ell \cdot \ell') \right) \left| \tilde{h}(\ell, \ell') \right|^2 \\
&= \beta \int d\ell \, d\ell' \ e^{-\frac{\beta}{2}(\frac{\ell^2}{4}+\ell'^2)} \, \frac{\sinh\left[\frac{\beta}{2}(\ell\cdot\ell')\right]}{\ell\cdot\ell'} \ \left| \tilde{h}(\ell, \ell') \right|^2 , \\
&= \frac{\beta}{(2\pi\lambda)^n} \int dx \, dp \ e^{-\beta p^2/2} \ h(x,p) \left[e^{-\frac{\beta}{8}\hat{p}^2} \ \frac{\sinh\left[\frac{\beta}{2}(\hat{p}\cdot p)\right]}{\hat{p}\cdot p} \ h(x,p) \right] , \\
&= \frac{\beta^{2-\frac{n}{2}}}{(2\pi\lambda^2)^{\frac{n}{2}}} \ll \check{h} \, \| \, \hat{\hat{E}}(\beta) \, \| \, \check{h} \gg , \qquad (60)
\end{aligned}
$$

where, on the third line, one kept only the even part in ℓ of the integrand (as it must be) and, on the fifth line, one made use of (50). The checked functions in (60) are defined by (51), and the effective kinetic operator by

$$
\begin{aligned}
\hat{\hat{E}}(\beta) \ &:= \ e^{-\frac{\beta}{8}\hat{p}^2} \ e^{\frac{1}{2}(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v})} \left(\frac{\sinh\left[\frac{1}{2}\beta^{\frac{1}{2}} \ (\hat{p}\cdot v)\right]}{\beta^{\frac{1}{2}}(\hat{p}\cdot v)} \right) \ e^{-\frac{1}{2}(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v})} \\
&= \ e^{-\frac{\beta}{8}\hat{p}^2} \ e^{\frac{1}{2}\left[\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v} \, , \quad \right]_-} \left(\frac{\sinh\left[\frac{1}{2}\beta^{\frac{1}{2}} \ (\hat{p}\cdot v)\right]}{\beta^{\frac{1}{2}}(\hat{p}\cdot v)} \right) \\
&= \ \hat{\hat{1}} - \frac{\beta}{12} \left(\hat{p}^2 - (\hat{p}\cdot v)(\hat{p}\cdot\frac{\partial}{\partial v}) - \frac{1}{2}(\hat{p}\cdot v)^2 - \frac{1}{2}(\hat{p}\cdot\frac{\partial}{\partial v})^2 \right) \\
&\qquad + \mathcal{O}(\beta^2) \, . \qquad (61)
\end{aligned}
$$

The kinetic operator of the quadratic part (60) in the effective action is a power series in β

$$
\hat{\hat{E}}(\beta) = \sum_{m\geqslant 0} \frac{1}{m!} \, \hat{\hat{E}}_n \, \beta^n
$$

whose coefficient $\hat{\hat{E}}_m$ is a differential operator of order $2m$. It should be emphasized that (61) does *not* starts with the local kinetic operator given by Frønsdal in [2] (even though, for spin-one, this term reproduces the Maxwell Lagrangian).

The part of the one-loop effective action

$$
\begin{aligned}
S_{eff}[g] - S_{eff}[h_0] \ &= \ \frac{1}{(2\pi\lambda^2)^{\frac{n}{2}}} \lim_{\Lambda\to\infty} \Big[\Lambda^{\frac{n}{2}-1} \int dx \, \check{h}(x) \\
&\qquad + \sum_{m=0}^{[\frac{n}{2}]-3} \Lambda^{\frac{n}{2}-3-m} \ll \check{h} \, \| \, \hat{\hat{E}}_m \, \| \, \check{h} \gg \Big] \\
&\qquad + \text{finite} + \mathcal{O}(h^3) \, ,
\end{aligned}
$$

that is divergent in the ultraviolet corresponds to the induced classical action since these terms should be eliminated through some renormalization of counterterms (Field renormalizations are already implicitly assumed since the field redefinition $h \to \check{h}$ has been performed).

7 Propagator residue

The *generating functional of Green functions* is defined as

$$
Z\,[\,J\,] := \int \mathcal{D}h \, e^{-S[\phi=0\,,h] - \ll h \, \| \, J \gg} \ \propto \ e^{-\frac{1}{2}\ll J \, \| \, \hat{P} \, \| \, J \gg + \mathcal{O}(J^3)} \qquad (62)
$$

where the operator $\hat{\hat{P}}$ is called the *propagator* and the quadratic form given by $\ll J \,\|\, \hat{\hat{P}} \,\|\, J \gg$ describes the free propagation of a source J . This comes from the expansion (29) with (28) and (52). The source[2] must obey to the transversality condition $(\hat{p}\cdot\partial/\partial v)\,J(x,v)=0$ at linearized order, in order for the generating functional (62) to be gauge invariant under (21), as follows from (26). The stationary points of the functional (62) are determined by

$$\hat{\hat{K}}\,h(x,v)+\mathcal{O}(h^2)=J(x,v)\quad\Longleftrightarrow\quad h(x,v)=\hat{\hat{P}}\,J(x,v)+\mathcal{O}(J^2)\,,\tag{63}$$

where the propagator $\hat{\hat{P}}$ is determined from (53) takes the form

$$\hat{\hat{P}}=\frac{\hat{R}}{\hat{p}^2}\,+\,(v\cdot\hat{p})\,\hat{\hat{T}}\tag{64}$$

and the operator $\hat{R}$ on the right-hand-side is defined (formally) as $\hat{R}:=\hat{M}^{-1}$. The transversality condition and the explicit form of the propagator (64) implies that

$$\ll J \,\|\, \hat{\hat{P}} \,\|\, J \gg\;=\;\ll J \,\|\, \frac{\hat{R}}{\hat{p}^2} \,\|\, J \gg\,,\tag{65}$$

so that the operator $\hat{R}(v,\partial/\partial v)$ is called the *residue of the propagator* on the gauge field mass-shell (*i.e.* the light-cone). A nice feature of the formula (65) is that the residue is a self-adjoint operator which acts only on the spin degrees of freedom, so the that the dependence of the quadratic form (65) on the momentum is simply in the denominator. Notice that, as (54), the residue is also a power series of the form

$$\hat{R}(v,\partial/\partial v)\;=\;\sum_{m\geqslant 0}\,R_m\Big(v\cdot\frac{\partial}{\partial v}\Big)\;\Big(v\cdot v\Big)^m\Big(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\Big)^m\,,\tag{66}$$

where the coefficients R_m are rational functions of a single variable. Consequently, the normal symbol $r(v,u):=R(y,z)$ of the residue is only a function of the two variables

$$y\;:=\;v\cdot u\,,\quad z\;=\;\frac{1}{4}(v\cdot v)(u\cdot u)\,.\tag{67}$$

Notice that y is the normal symbol of the spin operator $\hat{Y}$ defined in (23).

As is well known, on-shell the quadratic form $\ll h \,\|\, \hat{\hat{P}} \,\|\, h' \gg$ encodes the two-point free scattering amplitude between physical states. Therefore a crucial requirement on the residue of the propagator is that, when J is divergenceless, the quadratic form $\ll J \,\|\, \hat{R} \,\|\, J \gg$ must only contain components of J which are traceless and transverse to the light-cone. As can be checked explicitly by using (66), if the source J obeys to the transversality condition, then the quadratic form (65) is invariant on-shell under the transformations $J\to J+(v\cdot\hat{p})\,\varepsilon$ with transverse parameter $(\hat{p}\cdot\partial/\partial v)\,\varepsilon=0$ in order to preserve the transversality hypothesis. Therefore, in the quadratic form (65) the components of a divergenceless source can be assumed, on-shell, to be transverse to the light-cone without loss of generality. More concretely, in the light-cone coordinate system $x^\mu=(\,x^a\,,x^+,x^-)$ such that the momentum takes takes the form $k=(0,0,k_-)$, one has

$$\ll J \,\|\, \hat{R} \,\|\, J \gg\;\approx\;\ll j \,\|\, \hat{r} \,\|\, j \gg\,,$$

where

$$j\,(k^\mu,v^a)\;:=\;J(k^\mu,v^\nu)\Big|_{v^\pm=0}$$

are the components of J transverse to the light-cone and the operator $\hat{r}$ is the restriction of $\hat{R}$ on this subspace, *i.e.* in terms of their normal symbols $r\left(v^a,u^b\right):=r\left(v^a,v^\pm=0,u^b,u^\pm=0\right)$.

[2]In the present context, J does not physically correspond to conserved currents but is merely the argument of the functional (62), so it is only an auxiliary variable not a physical observable.

But the unphysical traceful components must also be absent from the quadratic form, thus the image of the operator $\hat{r}$ must be contained into the kernel of the $(n-2)$-dimensional trace operator which is transverse to the light cone. In other words, it must obey to

$$\Big(\frac{\partial}{\partial v^a}\frac{\partial}{\partial v_a}\Big)\hat{r} = 0\,. \tag{68}$$

The solution of (68) is not unique since any product $\hat{r}\,' = \hat{r}\,f(\hat{y})$ of a solution $\hat{r}$ with any function $f(\hat{y})$ of the "spin" operator $\hat{y} := v^a\frac{\partial}{\partial v^a}$ is also a solution of (68). This freedom corresponds to spin-dependent choices of normalizations of the tensor fields. The general solution $\hat{r}\,'$ is found by taking $\hat{r}$ to be equal to the $(n-2)$-dimensional projector on the harmonic functions of v^a. This corresponds to coefficients [8]

$$R_m(y) \;=\; \frac{f(y)}{4^m\,m!\,\prod\limits_{j=1}^{m}\big(j+2-\frac{n}{2}-y\big)} \;=\; \frac{f(y)\,B(m\,,3-\frac{n}{2}-y)}{4^m\,(m-1)!\;m!}\,,$$

for the ansatz (66). The Euler Beta function B is defined by

$$B(p,q) \;:=\; \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \;=\; \int\limits_0^1 dt\,(1-t)^{p-1}\,t^{q-1}\,,$$

when $\Re(p)>0$ and $\Re(q)>0$. Therefore, the residue can be rewritten as

$$\begin{aligned}\hat{R}(v,\partial/\partial v) = &\sum_{m\geqslant 0}\frac{1}{4^m\,(m-1)!\;m!}\;\times\\ &\times\Big(v\cdot v\Big)^m f\Big(\hat{Y}+2m\Big)B\Big(m,3-2m-\frac{n}{2}-\hat{Y}\Big)\Big(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}\Big)^m,\end{aligned} \tag{69}$$

since $\hat{Y}\big(v\cdot v\big) = \big(v\cdot v\big)(\hat{Y}+2)$. In order to compute the normal symbol of the residue, some preliminary results are needed. The (left) normal product of y with any function $g(v,u)$ is equal to

$$y\star_N g(u,v) \;=\; \left(y+v\cdot\frac{\partial}{\partial v}\right)g(u,v)\,,$$

thus for a function of y only

$$y\star_N g(y) \;=\; y\,\left(1+\frac{d}{dy}\right)g(y)\,, \tag{70}$$

therefore one can compute the normal symbol of any operator function of the "spin" operator (23) only,

$$\mathcal{N}^{-1}\Big[f(\hat{Y})\Big] = \mathcal{N}^{-1}\Big[f(\hat{Y})\,\hat{1}\Big] \;=\; f\,(\,y\star_N)\,1 \;=\; f\left(y\Big(1+\frac{d}{dy}\Big)\right)1\,. \tag{71}$$

In some particular cases such as the operator $\hat{F}_t\left(\hat{Y}\right) := t^{\hat{Y}}$ one may be even more explicit since it is the solution of the Cauchy problem

$$\left\{\begin{array}{l}\Big[\,t\frac{\partial}{\partial t}-\hat{Y}\Big]\hat{F}_t(\hat{Y}) = \hat{0}\\[2ex] \hat{F}_t(\hat{0}) = \hat{1}\end{array}\right. \quad\Longleftrightarrow\quad \left\{\begin{array}{l}\Big[\,t\frac{\partial}{\partial t}-y\left(1+\frac{\partial}{\partial y}\right)\Big]f_t(y) = 0\\[2ex] f_t(0) = 1\end{array}\right.\,,$$

where $f_t(y) := \mathcal{N}^{-1}\big[\,\hat{F}_t(\hat{Y})\big]$ is its normal symbol and use has been made of (70). The solution of this linear partial differential equation can be checked to be

$$\mathcal{N}^{-1}\big[\,t^{\hat{Y}}\,\big] \;=\; e^{(t-1)\,y} \tag{72}$$

Thus (71) and (72) imply that the normal symbol of the residue (69) is equal to the power series

$$r(v,u) = R(y,z) = \sum_{m\geqslant 0} \frac{1}{m!}\, r_m(y)\, z^m \tag{73}$$

with coefficients of the variable z introduced in (67) given by

$$r_m(y) := \frac{1}{(m-1)!}\, f\Big(\, (y+2m)\, \star_{_N}\,\Big) \int_0^1 dt\, (1-t)^{m-1}\, e^{(1-t)\,y}\, t^{2(1-m)-\frac{n}{2}}\,. \tag{74}$$

8 Current exchange interaction

The *matter effective action* $S_{eff}[\phi]$ that comes from the action (29) is defined by integrating out the gauge fields,

$$\int \mathcal{D}h\, e^{-S[\phi\,,h]} \,\propto\, e^{-S_{eff}[\phi]}\,. \tag{75}$$

This path integral can be evaluated analogously to the generating functional (62) of Green functions. Effectively, the Noether interaction leads to the *current exchange interaction* for the matter fields defined as

$$S_{curr}\,[\,\phi\,] \;:=\; \frac{1}{2}\;\ll J \,\|\, \frac{\hat{\hat{R}}}{\hat{p}^2} \,\|\, J \gg\;. \tag{76}$$

More precisely, at tree level and modulo field redefinitions, the matter effective action is equal to

$$S^{\,tree}_{eff}[\phi] \;=\; S_0[\phi] \,+\, S_{curr}[\phi] \,+\, \mathcal{O}(\phi^5)\,, \tag{77}$$

since the conserved currents J are quadratic in ϕ and since the weak equality $\ll J \,\|\, \hat{\hat{P}} \,\|\, J \gg \approx \ll J \,\|\, \frac{\hat{R}}{\hat{p}^2} \,\|\, J \gg$, which follows from (31) together with (64), corresponds to field redefinitions. Due to the relation (48), the current exchange interaction (76) reads in terms of the normal symbol $r(v,u) = R(y,z)$ of the propagator residue as

$$S_{curr}\,[\,\phi\,] \;=\; \int \frac{dk}{k^2}\;\, w\Big(\frac{\partial}{\partial v}\,,\,\frac{\partial}{\partial v'}\Big)\;\tilde{J}(k,v)\;\tilde{J}(-k,v')\Big|_{v=v'=0}\,, \tag{78}$$

where k is the exchanged momentum, $\tilde{J}(k,v)$ is the Fourier transform over spacetime of the current generating function and where the normal symbol $w(v,u) = W(y,z)$ is defined in terms of the normal symbol (73) as follows

$$w(v,u) = W(y,z) \;:=\; \frac{(2\pi\lambda)^n}{2}\;\exp(y)\;R(y,z) \tag{79}$$

and is only a function of the variables (67).

The matter effective action that comes from the action (39) where ϕ is a complex scalar field minimally coupled to an infinite tower of tensor gauge fields is equal, at tree level and modulo field redefinitions, to (77) where the quadratic functional S_0 is the Klein-Gordon action (33) while the quartic functional S_{curr} is the current exchange interaction (76). The latter can be explicitly computed in momentum space. Indeed, the cubic vertex itself takes a very simple form in momentum space in terms of the Fourier transforms over spacetime. Since

$$\phi(x - i\,\lambda\, v/2) \;=\; \int dk\;\tilde{\phi}(k)\; e^{\frac{i}{\lambda}\,k\,\cdot(x-i\,\lambda\,v/2)} \;=\; \int dk\;\tilde{\phi}(k)\; e^{\frac{i}{\lambda}\,k\,\cdot\,x}\; e^{\,k\,\cdot\,v/2}\,,$$

one finds

$$|\,\phi(x - i\,v/2)\,|^2 \;=\; \int dk_1 dk_2\; \tilde{\phi}(k_1)\,\tilde{\phi}^*(-k_2)\; e^{\frac{i}{\lambda}\,(k_1+k_2)\,\cdot\,x}\; e^{\,(k_1-k_2)\,\cdot\,v/2}\,,$$

thus the Fourier transform of the generating function (34) over spacetime is

$$\tilde{J}(k,v) \;=\; \int dk_1 dk_2\; \tilde{\phi}(k_1)\, \tilde{\phi}^*(-k_2)\; e^{(k_1-k_2)\cdot v/2}\, \delta(k_1+k_2-k)\,. \tag{80}$$

The Noether interaction (28) reads

$$\begin{aligned} S_1[\phi,h] \;&=\; (2\pi\lambda)^n \int dk\; \exp\left(\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v'}\right) \tilde{h}(k,v)\, \tilde{J}'(-k,v')\Big|_{v=v'=0}\,, \\ &= (2\pi)^n \int dk\, dk_1 dk_2\; \tilde{\phi}(k_1)\, \tilde{\phi}^*(-k_2)\; \delta(k+k_1+k_2)\,\times \\ &\qquad \times \exp\Big(\frac{k_1-k_2}{2}\cdot\frac{\partial}{\partial v}\Big)\tilde{h}(k,v)\Big|_{v=0}\,, \\ &= (2\pi)^n \int dk\, dk_1 dk_2\; \tilde{\phi}(k_1)\, \tilde{\phi}^*(-k_2)\; \delta(k+k_1+k_2)\, \tilde{h}\big(k,\frac{k_1-k_2}{2}\big). \end{aligned}$$

Alternatively, this result could have been obtained by inserting the completeness relations $\int dk \mid k\rangle\langle k\mid = \widehat{1}$ between each state in (37) and apply the identity (10). Inserting (80) into (78) leads to the current exchange interaction in the form

$$\begin{aligned} S_{curr}\,[\,\phi\,] \;&=\; \int dk_1\, dk_2\, dk_3\, dk_4\;\; \delta(k_1+k_2+k_3+k_4)\;\times \\ &\qquad \times\, \tilde{\phi}(k_1)\, \tilde{\phi}^*(-k_2)\, \tilde{\phi}(k_3)\, \tilde{\phi}^*(-k_4)\, A(k_1,k_2,k_3,k_4)\,, \end{aligned} \tag{81}$$

where the amplitude is given by

$$A(k_1,k_2,k_3,k_4) \;=\; \frac{w\left(\frac{k_1-k_2}{2}\,,\,\frac{k_3-k_4}{2}\right)}{(k_1+k_2)^2}\,, \tag{82}$$

with the function $w(v,u)$ defined in (79).

The *Mandlestam variables* of this four-particle elastic ($k_i^2=-m^2$) scattering are [14]

$$s=-(k_1+k_2)^2\,,\quad t=-(k_2+k_3)^2\,,\quad u=-(k_1+k_3)^2\,,$$

with the relation $s{+}t{+}u = 4\,m^2$. The amplitude (82) corresponds to an s-channel process $\phi{+}\overline{\phi}\to\phi{+}\overline{\phi}$ where s is the squared center of mass energy and t is the squared momentum transfer. The *scattering angle* in the center of mass system is determined by the relation

$$\cos\theta \;=\; 1+\frac{2\,t}{s-4\,m^2} \;=\; \frac{u-t}{u+t}\,.$$

The products of momenta are related by

$$k_1\cdot k_2 = k_3\cdot k_4 = m^2-\frac{s}{2}\,,\qquad k_2\cdot k_3 = k_1\cdot k_4 = m^2-\frac{t}{2}\,,$$

$$k_1\cdot k_3 = k_2\cdot k_4 = m^2-\frac{u}{2}\,.$$

In these variables, the amplitude (82) reads as

$$A(s,t,u) \;=\; -\,\frac{W\Big(t-u\,,\,\frac{1}{32}(t+u)^2\Big)}{s}\,, \tag{83}$$

where the function $W(y,z)$ is defined in (79) and its arguments in (67). Notice that if the scalar field is real then one can insert the relation $\tilde{\phi}^*(-k)=\phi(k)$ in (81), implying that only the term which is invariant under the exchange $t\leftrightarrow u$ (that is, even in y) should be kept in the amplitude (82).

At large energies, *i.e.* in the limit where the ratio $s/m^2 \to \infty$, the scattering angle θ is determined by

$$\cos\theta \sim 1 + \frac{2\,t}{s}$$

therefore the high-energy fixed-angle limit corresponds to s/m^2 large with $s/\,t$ fixed (thus $t/m^2 \to -\infty$). The amplitude (82) behaves as follows

$$A(s,t,u) \,\sim\, -\,\frac{W(s+2\,t\,,\,\frac{1}{32}\,s^2)}{s} \sim -\,\frac{W\big(\cos\theta\,s\,,\,\frac{1}{32}\,s^2\big)}{s}\,,\,, \tag{84}$$

where the asymptotic behaviour of the function $W(y,z)$ is unfortunately hard to estimate because its explicit expression is rather complicate, since the function $r(y,z)$ in the definition (79) is a power series (73) with coefficients (74) given by intricate integrals. Nevertheless, it would be interesting to compare its behaviour with the exponential fall-off of the ultraviolet fixed-angle Veneziano and Virasoro four-tachyon amplitudes.

9 Conclusion

As advocated here, the Noether procedure applied to an infinite tower of (higher-rank) conserved currents associated with (higher-derivative) symmetries of the Klein-Gordon equation is deeply connected with Weyl quantization and leads to a gauge symmetry group which is (at lowest order) isomorphic to the group of unitary operators on $\mathbb{R}^n$. Apart from technical complications, the straight analogue of this cubic coupling between a tower of (higher-spin) gauge fields and a free scalar field on any Riemannian manifold $\mathcal{M}$ should lead to the group of unitary operators on $\mathcal{M}$. The only difference would be that the Noether procedure could hold for homogeneous manifolds only, in order for conserved currents to exist. Apart from suggesting some nonAbelian symmetry group, the use of symbol calculus enabled to write the cubic vertex in a very compact form which allows an explicit computation of the four-scalar amplitude and of the effective actions at lowest ord ers. Unfortunately, although invariant under the non-Abelian gauge transformations, the gauge effective action is not a satisfactory physical candidate because it contains a linear term and, moreover, its quadratic term does not correspond to a proper free action.

The subtle issue of the trace constraints of Frønsdal [2] on the gauge fields and parameters in higher-spin metric-like theory has not been discussed in the previous sections and deserves some comments. These constraints might have been included by consistently imposing weaker conservation laws on double-traceless currents. Nevertheless, it was convenient to remove trace constraints when reflecting on the nonAbelian symmetry group. Moreover, the trace constraints may be removed in the action principle for free higher-spin metric-like fields itself in several ways (see [15] for some reviews, and [8, 16] for later developments). As far as the nonAbelian frame-like formulation is concerned, the straight analogue of Vasiliev's unfolded equations in the unconstrained case is dynamically empty and can somehow be thought of as Fedosov's quantization [6]. But a slight modification of the former has been proposed in [17] and should be dynam ically interesting. Last but not least, the group of symmetries of the metric-like theory arising from these unconstrained frame-like theories (by fixing the gauge and solving the torsion constraints) can be shown to be also isomorphic to the group of unitary operators on $\mathbb{R}^n$, at lowest order in the gauge fields and around flat spacetime [18].

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